

The Axisymmetric Case for the Post-Newtonian Dedekind Ellipsoids

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ABSTRACT

We consider the post-Newtonian approximation for the Dedekind ellipsoids in the case of axisymmetry. The approach taken by Chandrasekhar & Elbert (1974, 1978) excludes the possibility of finding a uniformly rotating (deformed) spheroid in the axially symmetric limit, though the solution exists at the point of axisymmetry. We consider an extension to their work that permits the possibility of such a limit.

Subject headings: Dedekind ellipsoids, non-axisymmetric, stationary

1. Introduction

The Dedekind tri-axial ellipsoids are an example of non-axisymmetric, but stationary solutions within Newtonian gravity. Due to internal motions, they are, in fact, stationary in an inertial frame. When addressing the question of whether or not stationary, but non-axisymmetric solutions are possible within General Relativity, this property makes the Dedekind ellipsoids a natural choice upon which to base one's considerations. It was, in part, with this question in mind that Chandrasekhar & Elbert (1974, 1978) turned their attentions to the post-Newtonian (PN) approximation of the Dedekind ellipsoids. In a paper from the same series, Chandrasekhar (1967b) had already considered the axisymmetric limit of the PN Jacobi ellipsoids at length and was able to show that it coincides with a certain PN Maclaurin spheroid (just as their Newtonian counterparts coincide at the point of bifurcation). This is related to the fact that the PN figures were chosen to rotate uniformly. On the other hand, the PN velocity field chosen in Chandrasekhar & Elbert

(1978) excludes the possibility of uniform rotation in the axisymmetric *limit* although it is possible in the axisymmetric case. This restriction seems neither natural nor advisable in the context of trying to settle the question as to the existence of relativistic, non-axisymmetric, stationary solutions. The naïve expectation is that the axisymmetric PN Dedekind ellipsoids contain the PN Maclaurin spheroids in the axisymmetric limit (up to arbitrary order).

In this article, we begin in § 2 by examining the axisymmetric case of a generalization to the solution presented in Chandrasekhar & Elbert (1978). We proceed in § 3 to consider a (continuous) limit to axisymmetry. In § 4 the connection to the post-Newtonian Maclaurin spheroids is examined.

2. The Axisymmetric Solution of a Generalization to Chandrasekhar and Elbert's Paper

We consider a generalization of the PN Dedekind ellipsoids presented in Chandrasekhar & Elbert (1978) (referred to from here on in as CE78) in

which we add post-Newtonian terms to the velocity. We comply with the notation used in CE78 and refer the reader to the definitions there for the various quantities. The post-Newtonian contributions to the velocity, which we introduce here are

$$\begin{aligned}\delta v_1 &= a_1^2 w_1 x_2 + (q_1 + q) x_1^2 x_2 + r_1 x_2^3 + t_1 x_2 x_3^2 \\ \delta v_2 &= a_2^2 w_2 x_1 + (q_2 - q) x_1 x_2^2 + r_2 x_1^3 + t_2 x_1 x_3^2 \\ \delta v_3 &= q_3 x_1 x_2 x_3,\end{aligned}\tag{1}$$

where the terms with w_1 and w_2 have been added for reasons that will be made clear when we discuss the solution. Note that we could eliminate one constant by introducing variables to denote $q_1 + q$ and $q_2 - q$, but choose instead to retain the notation in CE78.¹

The Newtonian ellipsoid is characterized by the semi-major axes $a_1 \geq a_2 \geq a_3$. Let us assume for the moment that, as in the Newtonian setting, the axisymmetric case is obtained by considering $a_2 = a_1$, an assumption that will be verified shortly. In this case, the index ‘2’ in the index symbols $A_{ijk\dots}$ and $B_{ijk\dots}$ used in CE78 and discussed at length in § 21 of Chandrasekhar (1987) can be replaced by ‘1’ as is evident from the definitions. Using the relations given in that book, it is possible to reduce all the index symbols to A_1 and A_2 . At the point $a_2 = a_1$, the value for A_1 (and thus A_2) is given by (36) in § 17 of Chandrasekhar (1987). Furthermore, (2) from Chandrasekhar & Elbert (1974) shows us that

$$Q_2 \stackrel{a}{=} -Q_1,\tag{2}$$

where we define the symbol $\stackrel{a}{=}$ to mean that the expression is evaluated at the point $a_2 = a_1$, i.e.

$$C|_{a_2=a_1} = D|_{a_2=a_1} \text{ is denoted by } C \stackrel{a}{=} D.\tag{3}$$

The value for a_3 can be found from the equation

$$a_1^2 a_2^2 A_{12} = a_3^2 A_3,\tag{4}$$

which holds for the Dedekind (and Jacobi) ellipsoids, and gives the value

$$\frac{a_3}{a_1} \stackrel{a}{=} 0.5827241661\dots\tag{5}$$

¹The three-velocity v^i in CE78 does not refer to the spatial components of the four-velocity $u^\alpha = dx^\alpha/d\tau$, but is instead defined as $v^i = dx^i/dt = u^i/u^0$.

Throughout this paper, a_3 is to be understood as a function of a_1 and a_2 , given by (4).

We can now consider the integrability conditions for the pressure and the continuity equation. We again follow CE78 and shall refer to the equation numbers there by adding a prime. It turns out that (38') (of CE78) remains unchanged despite the modification to the velocity, so that we find

$$q_3 \stackrel{a}{=} 0\tag{6}$$

and then from (24') that

$$q_2 \stackrel{a}{=} -q_1.\tag{7}$$

Equation (28') is identically fulfilled for $a_2 = a_1$, meaning that q_1 is left undetermined, in contrast to the general case.

With the changes to the velocity, equations (30') and (31') gain the additional terms $(a_1^2 Q_2 w_1 + a_2^2 Q_1 w_2) x_1$ and $(a_1^2 Q_2 w_1 + a_2^2 Q_1 w_2) x_2$ respectively. Equations (32')–(38') remain unchanged. Equation (32') yields

$$r_2 \stackrel{a}{=} -r_1\tag{8}$$

and (37') gives

$$t_2 \stackrel{a}{=} t_1\tag{9}$$

(we shall see shortly that each t_i becomes zero). There are additional terms in (39') corresponding to adding $-(a_1^2 Q_2 w_1 + a_2^2 Q_1 w_2)/2 = a_2^2 Q_1 (w_1 - w_2)/2$ both to α_1^{78} and α_2^{78} .

Requiring for the new velocity that its normal component vanish on the surface leads to a change in (50') and thus the resulting equations (52')–(56') by which the terms with $S_1 - S_2$ are modified. They now become

$$S_1 - S_2 - \frac{w_1 + w_2}{2Q_1}.\tag{10}$$

Using equations (2), (7) and (8), we can subtract equation (54') from (55') in CE78 to arrive at

$$q + q_1 - r_1 \stackrel{a}{=} \frac{4}{3} Q_1 (4S_3 + S_4).\tag{11}$$

²We use the superscripts ‘67’ and ‘78’ to distinguish the quantities defined in Chandrasekhar (1967b) from those in Chandrasekhar & Elbert (1978).

Next we turn our attention to the system of equations (58') from CE78³. In the case being considered here, the first of these equations becomes

$$\begin{aligned}
0 &= Q_{11}^{78} + \frac{a_2^4}{a_1^4} Q_{22}^{78} - \frac{a_2^2}{a_1^2} Q_{12}^{78} \\
&\stackrel{a}{=} P_{11}^{78} + P_{22}^{78} - P_{12}^{78} - \frac{2a_3^2}{3a_1^2} A_3(4S_3 + S_4) \\
&= \alpha_{11}^{78} + \alpha_{22}^{78} - \alpha_{12}^{78} + Q_1(q - r_1) \\
&\quad - \frac{2a_3^2}{3a_1^2} A_3(4S_3 + S_4) \\
&\quad + \sum_{i=3}^5 S_i \left(u_{11}^{(i)} + u_{22}^{(i)} - u_{12}^{(i)} \right) \\
&= \frac{2}{3} Q_1^2(4S_3 + S_4) - \frac{2a_3^2}{3a_1^2} A_3(4S_3 + S_4) \\
&\quad + \frac{2}{3} a_1^6 A_{1111}(4S_3 + S_4), \tag{12}
\end{aligned}$$

where the values for the α 's and their axisymmetric limits can be found in Appendix A, the u 's are given in Chandrasekhar (1967b) (C67b)⁴ equations (72) and (73) and where we made use of (11) from the current paper. The unique solution to this equation is

$$S_4 \stackrel{a}{=} -4S_3 \tag{13}$$

as it is for the analogue equation (100) of C67b despite the fact that the term with $q - r_1$ is absent there. With the result (13), (11) becomes

$$q + q_1 - r_1 \stackrel{a}{=} 0 \tag{14}$$

and for equation (53') from CE78⁵, or equivalently the sum of (54') and (55'), we find

$$S_1 - S_2 \stackrel{a}{=} -\frac{5}{3} S_3. \tag{15}$$

³Please note that we have been unable to reproduce the values from Table 1' in CE78 that result from solving (58'). A detailed discussion can be found in Appendix A.

⁴As mentioned in C67b, the u 's belonging to the displacements $\xi^{(4)}$ and $\xi^{(5)}$ are generated by cyclically permuting the indices. The precise meaning is best understood via the example that $u_1^{(4)}$ can be generated from $u_3^{(3)} = -\frac{1}{2} a_1^2 (a_2^2 B_{123} - a_1^2 B_{113})$ and becomes $u_1^{(4)} = -\frac{1}{2} a_2^2 (a_3^2 B_{231} - a_2^2 B_{221})$.

⁵In (53') of CE78, the factor Q_1 is missing from the term with $(S_1 - S_2)$.

The third minus the second of equations (58') is the analogue of equation (101) in C67b and is in fact precisely the same equation despite the different definitions for P_{ij} :

$$\begin{aligned}
0 &= Q_{11}^{78} - \frac{a_2^4}{a_1^4} Q_{22}^{78} - \frac{a_3^2}{a_1^2} Q_{31}^{78} - \frac{a_2^2 a_3^2}{a_1^4} Q_{23}^{78} \\
&\stackrel{a}{=} P_{11}^{78} - P_{22}^{78} - \frac{a_3^2}{a_1^2} (P_{31}^{78} - P_{23}^{78}) \\
&\quad - 2 \frac{a_3^2}{a_1^4} A_3 \left(\frac{17}{3} a_1^2 S_3 + a_3^2 S_5 \right) \\
&= \alpha_{11}^{78} - \alpha_{22}^{78} - \frac{a_3^2}{a_1^2} (\alpha_{31}^{78} - \alpha_{23}^{78}) \\
&\quad - 2 \frac{a_3^2}{a_1^4} A_3 \left(\frac{17}{3} a_1^2 S_3 + a_3^2 S_5 \right) \\
&\quad + \sum_{i=3}^5 S_i \left[\left(u_{11}^{(i)} - u_{22}^{(i)} \right) - \frac{a_3^2}{a_1^2} \left(u_{31}^{(i)} - u_{23}^{(i)} \right) \right] \\
&= -2 \frac{a_3^2}{a_1^4} A_3 \left(\frac{17}{3} a_1^2 S_3 + a_3^2 S_5 \right) \\
&\quad + (a_1^2 A_{111} + 6a_3^2 A_{113} - 7a_3^2 A_{1113}) \\
&\quad \times \left(\frac{17}{3} a_1^2 S_3 + a_3^2 S_5 \right). \tag{16}
\end{aligned}$$

The unique solution to this equation is

$$S_5 \stackrel{a}{=} -\frac{17a_1^2}{3a_3^2} S_3. \tag{17}$$

We can use (56') together with (9), (13), (15) and (17) to conclude that

$$t_1 \stackrel{a}{=} t_2 \stackrel{a}{=} 0. \tag{18}$$

Equation (47') of CE78 tells us that the bounding surface is axisymmetric to the first PN order if and only if (13), (15) and (17) hold. The PN velocity field of CE78 can then be seen to be axisymmetric in the limit we are discussing, when we additionally require

$$w_2 \stackrel{a}{=} -w_1. \tag{19}$$

Using what has been shown above, the third equation of (58') in CE78 can be used to find the value of S_3 (where the relationship between the α^{78} 's and the α^{67} 's can be found in Appendix A.1)

$$\begin{aligned}
0 &= a_1^4 Q_{11}^{78} + a_3^4 Q_{33}^{78} - a_1^2 a_3^2 Q_{31}^{78} \\
&\stackrel{a}{=} a_1^4 P_{11}^{78} + a_3^4 P_{33}^{78} - a_1^2 a_3^2 P_{31}^{78} + \frac{130}{9} a_1^2 a_3^2 A_3 S_3
\end{aligned}$$

$$\begin{aligned}
&= a_1^4 \alpha_{11}^{78} + a_3^4 \alpha_{33}^{78} - a_1^2 a_3^2 \alpha_{31}^{78} \\
&\quad + \frac{a_1^4 Q_1}{4} (q + r_1) + \frac{130}{9} a_1^2 a_3^2 A_3 S_3 \\
&\quad + \sum_{i=3}^5 S_i \left(a_1^4 u_{11}^{(i)} + a_3^4 u_{33}^{(i)} - a_1^2 a_3^2 u_{31}^{(i)} \right) \\
&= a_1^4 \alpha_{11}^{67} + a_3^4 \alpha_{33}^{67} - a_1^2 a_3^2 \alpha_{31}^{67} \\
&\quad + \frac{a_1^4 Q_1}{2} r_1 + \frac{130}{9} a_1^2 a_3^2 A_3 S_3 \\
&\quad + \sum_{i=3}^5 S_i \left(a_1^4 u_{11}^{(i)} + a_3^4 u_{33}^{(i)} - a_1^2 a_3^2 u_{31}^{(i)} \right). \tag{20}
\end{aligned}$$

The solution for S_3 is then

$$S_3 \stackrel{a}{\approx} -0.01742648312 + 0.1061462885 r_1, \tag{21}$$

the analytic expression of which can be found in Appendix (B).

We now turn to the fifth of equations (58') to solve for S_1 . The equation is

$$\begin{aligned}
0 &= a_3^4 Q_{33}^{78} - a_1^4 Q_{11}^{78} + a_3^2 \bar{Q}_3^{78} - a_1^2 \bar{Q}_1^{78} \\
&\stackrel{a}{=} a_3^4 P_{33}^{78} - a_1^4 P_{11}^{78} + a_3^2 P_3^{78} - a_1^2 P_1^{78} \\
&\quad + 2a_1^2 a_3^2 (3S_1 + \frac{35}{9} S_3) \\
&= a_3^4 \alpha_{33}^{67} - a_1^4 \alpha_{11}^{67} + a_3^2 \alpha_3^{67} - a_1^2 \alpha_1^{67} - \frac{a_1^4 Q_1}{2} r_1 \\
&\quad + 2a_1^2 a_3^2 (3S_1 + \frac{35}{9} S_3) - a_1^4 Q_1 w_1 \\
&\quad + \sum_{i=1}^2 a_1^2 S_i \left(a_3^2 u_3^{(i)} - a_1^2 u_1^{(i)} \right) \\
&\quad + \sum_{i=3}^5 S_i \left(a_3^4 u_{33}^{(i)} - a_1^4 u_{11}^{(i)} + a_3^2 u_3^{(i)} - a_1^2 u_1^{(i)} \right) \tag{22}
\end{aligned}$$

and the solution is

$$\begin{aligned}
S_1 &\stackrel{a}{\approx} -(0.2836731908 + 0.7419729757 r_1 \\
&\quad + 1.121542227 w_1), \tag{23}
\end{aligned}$$

cf. Appendix (B) for the analytic expression. The fourth equation is then identically fulfilled. We have obtained a solution to all the equations at the point $a_2 = a_1$ and have two remaining constants, w_1 and r_1 (although q and q_1 are not determined, they always appear in the combination $q + q_1$, which is equal to r_1 , cf. (14)).

3. The Axisymmetric Limit of a Generalization to Chandrasekhar and Elbert's Paper

Before discussing the solution obtained above, we consider the solution to the post-Newtonian equations not at the point $a_2 = a_1$, but in the limit $a_2 \rightarrow a_1$. The equations listed above, are also obtained as limiting relations. However, in the limit, we also obtain two new equations, one of which allows us to determine $\lim_{a_2 \rightarrow a_1} q_1$ and the other, say $\lim_{a_2 \rightarrow a_1} r_1$.

Equations (24'), (28') and (38') of CE78 provide a system of three linear equations for the quantities q_1 , q_2 and q_3 . After solving this linear system, the limit $a_2 \rightarrow a_1$ can be taken to give

$$\begin{aligned}
q_1 &\rightarrow -6\sqrt{2B_{11}} \left(4a_1^2 B_{111} + \frac{a_1^4}{a_3^2} B_{113} \right) \\
&= \frac{-(2e^2 + 1)^2 Q_1^3}{e^4} \\
&\approx 2.827158725, \tag{24}
\end{aligned}$$

where we have defined the eccentricity

$$e := \sqrt{1 - a_3^2/a_1^2} \tag{25}$$

and where the explicit expression for Q_1 is

$$\lim_{a_2 \rightarrow a_1} Q_1 = \lim_{a_2 \rightarrow a_1} -\sqrt{\frac{8e^2(1 - e^2)}{3 + 8e^2 - 8e^4}} \tag{26}$$

(we remind the reader that a_3 depends on a_1 and a_2 via (4)).

The fourth of equations (58') is identically fulfilled for $a_2 = a_1$. Therefore, we introduce

$$\varepsilon := 1 - a_2^2/a_1^2 \tag{27}$$

and expand the quantities involved and solve to first order in ε to give

$$\lim_{a_2 \rightarrow a_1} r_1 \approx 0.02880590648 - 1.75 \lim_{a_2 \rightarrow a_1} w_1. \tag{28}$$

We provide the analytic expression in Appendix B.

Strictly speaking, we have to show that the fourth of equations (58') is fulfilled to all orders in ε to be certain that (28) is continuously connected to the PN Dedekind solutions. We were able to solve the whole system of equations along the post-Newtonian Dedekind sequence for arbitrary w_1 and w_2 , meaning that the limit presented here can be tacked on continuously.

4. Discussion

The axially symmetric post-Newtonian solutions we have generated depend on two parameters or one if we require that the solution be continuously connected to the PN Dedekind ‘ellipsoids’ with the velocity field (1). The solutions are not uniformly rotating in general. If we add this constraint, then requiring that the four-velocity be shear-free tells us that

$$r_1 \rightarrow 0 \quad (\text{shear free}) \quad (29)$$

must hold.

We now show that with this additional constraint, the solution is indeed the PN Maclaurin solution (thereby demonstrating that the shear free condition is not only necessary, but also sufficient for uniform rotation in our case). Let us first note that upon taking into account the results above and in particular $Q_2 \rightarrow -Q_1$, the components of the velocity become

$$\begin{aligned} v_1 &= \sqrt{\pi G \rho} \left(Q_1 + \frac{\pi G \rho}{c^2} a_1^2 w_1 \right) x_2 \\ v_2 &= -\sqrt{\pi G \rho} \left(Q_1 + \frac{\pi G \rho}{c^2} a_1^2 w_1 \right) x_1 \\ v_3 &= 0. \end{aligned} \quad (30)$$

This is precisely the form of the velocity for the post-Newtonian Maclaurin spheroids, as can be found in Chandrasekhar (1967a) (C67a) equation (3), where Ω is a constant containing a Newtonian and post-Newtonian contribution, cf. (28) of that paper.

Next we note, that for a given equation of state, an axially symmetric, stationary and uniformly rotating fluid is described by two parameters. For our purposes, we can take them to be a_3/a_1 , which we prescribe using (4), and the value for a_1 , which we leave undetermined.

One has two additional degrees of freedom, which amount to the mapping between a Newtonian and post-Newtonian solution and is a matter of convention (cf. Bardeen 1971). For example, one can write the coordinate volume of the star to be

$$V = V_0 + V_1 \delta + \dots, \quad (31)$$

where δ is some relativistic parameter, and then choose to have the PN contribution vanish, $V_1 = 0$.

This is the choice that was made in CE78 and C67b and also in Chandrasekhar’s original paper on the PN Maclaurin spheroids C67a. We have followed this convention in the current paper, making it easy to compare our results to those of C67a. The second degree of freedom one has was left unspecified in much of C67a, though Table I lists values with the choice $S_1^M = S_3^M = 0$.⁶

If we introduce the new coordinate

$$\varpi^2 := x_1^2 + x_2^2, \quad (32)$$

and make use of (13), (15) and (17), then the bounding surface (cf. (47) in CE78) is given by

$$\begin{aligned} 0 &= \frac{\varpi^2}{a_1^2} + \frac{x_3^2}{a_3^2} - 1 - \frac{2\pi G \rho}{c^2} \left\{ S_1 \left(\varpi^2 - \frac{2a_1^2 x_3^2}{a_3^2} \right) \right. \\ &\quad + S_3 \left[\frac{5}{3} \left(\varpi^2 - \frac{a_1^2 x_3^2}{a_3^2} \right) - \frac{4}{3} \frac{\varpi^4}{a_1^2} + 4\varpi^2 \frac{x_3^2}{a_3^2} \right. \\ &\quad \left. \left. - \frac{17}{9} \frac{a_1^2 x_3^4}{a_3^4} + \frac{5}{3} x_1^2 \left(\frac{\varpi^2}{a_1^2} + \frac{x_3^2}{a_3^2} - 1 \right) \right] \right\}. \end{aligned} \quad (33)$$

Using the equation for the surface $\varpi^2/a_1^2 = 1 - x_3^2/a_3^2$, which holds at the Newtonian level and can thus be inserted into the PN term above, one sees that the term with x_1^2 vanishes and one finds that the equation is identical to (42) of C67a if

$$S_3 = -\frac{9}{13} S_2^M + \frac{3a_3^2}{13a_1^2} S_3^M \quad \text{and} \quad (34)$$

$$S_1 = S_1^M + \frac{16}{13} S_2^M - \frac{a_3^2}{13a_1^2} S_3^M \quad (35)$$

hold. As mentioned in that paper, $S_3^M = 0$ may be chosen without loss of generality⁷ which then leads to a unique relationship between S_3 and S_2^M , which is shown to be correct in Appendix B3. The constant S_1^M can be chosen arbitrarily just as with S_1 (which depends on w_1).

If one considers the *limit* $a_2 \rightarrow a_1$ and simultaneously requires that the star rotate uniformly,

⁶Where necessary, we distinguish the constants of C67a from those used here by adding the superscript ‘M’.

⁷Note that (34) and (35) together with (15) are equivalent to the three equations (123) of C67b as can be seen either by taking $S_3^M = 0$ or identifying α of that equation with $S_1^M + \frac{a_3^2}{3a_1^2} S_3^M$ and β with $S_2^M - \frac{a_3^2}{3a_1^2} S_3^M$.

then (28) provides the unique value for w_1 ,

$$w_1 \approx 0.01646051799, \quad (36)$$

which is equivalent to making a choice for S_1^M different from the one made in C67a, but no more and no less physically meaningful.

The most significant result of the analysis of the axisymmetric limit is that (28) shows us that the rigidly rotating limit ($r_1 = 0$) and the original choice of velocity field in CE78 ($w_1 = w_2 = 0$) are incompatible. While it is possible with that velocity field to find the post-Newtonian Maclaurin solution at the bifurcation point, this solution is not continuously connected to any other solution. When considering the question of the existence or non-existence of non-axially symmetric but stationary solutions, it seems important to retain the possibility of studying a neighbourhood of the axially symmetric and uniformly rotating limit, especially since such solutions are known to exist⁸. This possibility was excluded by the approach taken in CE78.

In a follow-up paper, we intend to tackle the problem with a more general approach that lends itself better to proceeding to higher post-Newtonian orders, is not as restrictive in the solutions it permits and allows one to show that the singularity discussed in CE78 is an artefact of the specific method chosen and not an inherent property of the post-Newtonian Dedekind solutions (cf. Gürlebeck & Petroff 2010).

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⁸As far as we know, there exists no formal proof demonstrating the existence of such solutions. Steps in that direction were taken by Heilig (1995) and the existence has been demonstrated by many groups that are able to solve Einstein’s equations numerically to extremely high accuracy, see e.g. Ansorg et al. (2003)

A. A Detailed Discussion of Chandrasekhar and Elbert's Work

We mentioned in footnote 3 that we have been unable to reproduce the values from Table 1' in CE78 that result from solving (58') nor have we succeeded in finding the source of the discrepancy. It is important to rule out an error in our understanding of that paper or an error in our own solutions to the equations presented there, and we therefore provide a detailed discussion here (in this section we use the velocity field in that paper, i.e. $w_1 = w_2 = 0$).

The calculations we performed were done with the aid of computer algebra. As a test, we did all the calculations using both Maple and Mathematica. To be absolutely certain that we solved the equations correctly, we wrote down the line element and energy-momentum tensor as given in Chandrasekhar (1965), had Mathematica (TTC package) determine Einstein's equations to first post-Newtonian order and then verified that they are indeed fulfilled. When the values from Table 1' of CE78 are inserted, then one finds that the condition that pressure vanish on the surface is violated at a level three orders of magnitude higher than with the values from our Table 1. We also verified that the violation vanishes in our case as more significant figures are added.

The solutions we found for q_1 , q_2 and q_3 agree with those given in Table 1' of CE78. This provides strong evidence suggesting that our numerical evaluation of a_3/a_1 for a given a_2/a_1 and of the index symbols is correct. Moreover, the dependence of q , r_1 , r_2 , t_1 and t_2 on S_i as given in equations (37') and (53')–(56') can be seen to hold both in Table 1' and Table 1. This indicates strongly that the typo in equation (53') of CE78 mentioned in footnote 5 is truly only that and that the quantities in the integrability condition of (11') are treated correctly in both papers, leaving only δU and Φ to be verified.

The system of linear equations providing the values for S_i , i.e. (58'), can of course be written as follows:

$$\begin{pmatrix} M_{11} & \cdots & M_{15} \\ \vdots & \ddots & \vdots \\ M_{51} & \cdots & M_{55} \end{pmatrix} \begin{pmatrix} S_1 \\ \vdots \\ S_5 \end{pmatrix} = \begin{pmatrix} N_1 \\ \vdots \\ N_5 \end{pmatrix}. \quad (\text{A1})$$

For a given value of a_2/a_1 , the matrix (M_{ij}) depends on the u 's from C67b and via their S_i dependence, indirectly on q , r_1 , r_2 , t_1 and t_2 . The vector (N_i) depends on the α 's and again on the (non- S_i dependent part of) q , r_1 , r_2 , t_1 and t_2 . We return to a discussion of this equation after mentioning a few incongruities in CE78.

In (44') a factor $1/(\pi G \rho)$ is missing in δU because the equation is copied directly from (74) of C67b, whereas the relationship between p/ρ and δU is not the same in (39') of CE78 and (75) of C67b. This mistake is corrected in (45') and (46') however. In (39') there is also a factor $1/(\pi G \rho)^2$ missing in the term $2\Phi + 2v^2U + \frac{1}{2} \left(\frac{p}{\rho} \right)^2$ as can be seen by checking dimensions⁹ and comparing to (11) in Chandrasekhar & Elbert (1974). Finally, we note that (A1) from above only ensures that the pressure is constant on the surface of the PN-ellipsoid as discussed in C67b, cf. (75) in *loc. cit.*, but not that it vanishes. The constant that would have to be determined to ensure vanishing pressure was not written in (39') or (40')¹⁰ and the constant that is a part of δU in (44') was dropped when proceeding to (45'). Since the determination of this constant plays no role in the paper however, we need not discuss it further and have not done so in our own paper.

We find that the determinant of (M_{ij}) vanishes at $a_2/a_1 = 0.33700003168 \dots$ just as in CE78, where it is given to four significant figures. This provides evidence suggesting that the matrices agree (and thus the δU) and that the vectors (N_i) disagree. If we multiply δU by a factor π , as suggested in the last paragraph,

⁹We advise the reader that, as mentioned after (14) in Chandrasekhar & Elbert (1974), the units in which Q_1 and Q_2 are measured change as of this point by a factor $\sqrt{\pi G \rho}$.

¹⁰The constant contained in δU is completely determined by (44') and is thus not available as a variable to ensure that the pressure vanish on the surface.

then the determinant becomes zero for $a_2/a_1 = 0.30874\dots$. Nonetheless, we tested that neither an arbitrary factor in front of this term, nor one in front of the term $2\Phi + 2v^2U + \frac{1}{2}\left(\frac{p}{\rho}\right)^2$ can explain the results in CE78.

A natural explanation for a disagreement between the vectors (N_i) in our case and in CE78 would be that one of the α 's contains a mistake. We checked to see that an arbitrary change in a single α cannot account for the differences in the results however. Since an explicit expression for these α 's is not provided in CE78, we cannot test directly to see whether or not each agrees. However, in the implicit expressions from (39') and (40'), only the contributions from $2\Phi + 2v^2U + \frac{1}{2}\left(\frac{p}{\rho}\right)^2$ are not written out. These can easily be compared to those written out explicitly for the α 's of C67b, where the appropriate modifications for the different Newtonian velocity have to be taken into account, and show perfect agreement with our expressions. In particular, the relationship to the α 's of C67b for $a_2 = a_1$, which is discussed in Appendix A.1 provides additional evidence for the correctness of our expressions. We also generated the α 's with computer algebra by typing out the expressions for (11'), solving the integrability condition and integrating it and showed that these agree with the expressions provided below.¹¹

$$\alpha_1^{78} = -\frac{a_3^4}{a_1^2}A_3^2 - 4a_2^2B_{12}(A_1 + A_2) - 2IQ_1Q_2 - (2I + 3a_3^2A_3)A_1 \\ + (a_1^2A_{11} - \frac{1}{2}B_{11})(2a_1^2Q_2^2 - 2a_1^2A_1 - 3a_3^2A_3) - \frac{1}{2}B_{12}(2a_2^2Q_1^2 - 2a_2^2A_2 - 3a_3^2A_3) + \frac{5}{2}a_3^2A_3B_{13} \quad (\text{A2})$$

$$\alpha_2^{78} = -\frac{a_3^4}{a_2^2}A_3^2 - 4a_1^2B_{12}(A_1 + A_2) - 2IQ_1Q_2 - (2I + 3a_3^2A_3)A_2 \\ + (a_2^2A_{22} - \frac{1}{2}B_{22})(2a_2^2Q_1^2 - 2a_2^2A_2 - 3a_3^2A_3) - \frac{1}{2}B_{12}(2a_1^2Q_2^2 - 2a_1^2A_1 - 3a_3^2A_3) + \frac{5}{2}a_3^2A_3B_{23} \quad (\text{A3})$$

$$\alpha_3^{78} = -a_3^2A_3^2 - (2I + 3a_3^2A_3)A_3 - \frac{1}{2}B_{23}(2a_2^2Q_1^2 - 2a_2^2A_2 - 3a_3^2A_3) \\ - \frac{1}{2}B_{13}(2a_1^2Q_2^2 - 2a_1^2A_1 - 3a_3^2A_3) + \frac{5}{2}a_3^2A_3(B_{33} - 2a_3^2A_{33}) \quad (\text{A4})$$

$$\alpha_{12}^{78} = \frac{a_3^4}{a_1^2a_2^2}A_3^2 - 2Q_1^2\left(A_1 + \frac{a_2^4}{a_1^4}A_2\right) + (2a_1^2Q_2^2 - 2a_1^2A_1 - 3a_3^2A_3)(-a_1^2A_{112} + \frac{1}{2}B_{112}) \\ + (2a_2^2Q_1^2 - 2a_2^2A_2 - 3a_3^2A_3)(-a_2^2A_{122} + \frac{1}{2}B_{122}) - \frac{5}{2}a_3^2A_3B_{123} + 2Q_1Q_2\left(1 - \frac{a_2^2}{a_1^2}\right)A_2 \\ - \frac{1}{2}Q_1^3Q_2 - 2a_2^2Q_1Q_2(3A_{22} + A_{12}) + 4Q_1^2\left(A_1 - \frac{a_2^2}{a_1^2}A_2\right) - Q_1(q_1 + \frac{1}{2}q_2) \quad (\text{A5})$$

$$\alpha_{23}^{78} = \frac{a_3^2}{a_2^2}A_3^2 - 2Q_1^2A_3 + 2Q_1^2\left(1 - \frac{a_2^2}{a_1^2}\right)A_3 - 2a_1^2Q_1Q_2(A_{13} + A_{23}) + (2a_1^2Q_2^2 - 2a_1^2A_1 - 3a_3^2A_3)(\frac{1}{2}B_{123}) \\ + (2a_2^2Q_1^2 - 2a_2^2A_2 - 3a_3^2A_3)(-a_2^2A_{223} + \frac{1}{2}B_{223}) - \frac{5}{2}a_3^2A_3(-2a_3^2A_{233} + B_{233}) \quad (\text{A6})$$

$$\alpha_{31}^{78} = \frac{a_3^2}{a_1^2}A_3^2 - 2Q_2^2A_3 + 2Q_1Q_2\left(1 - \frac{a_2^2}{a_1^2}\right)A_3 - 2a_2^2Q_1Q_2(A_{13} + A_{23}) - \frac{5}{2}a_3^2A_3(-2a_3^2A_{133} + B_{133}) \\ + (2a_1^2Q_2^2 - 2a_1^2A_1 - 3a_3^2A_3)(-a_1^2A_{113} + \frac{1}{2}B_{113}) + (2a_2^2Q_1^2 - 2a_2^2A_2 - 3a_3^2A_3)(\frac{1}{2}B_{123}) \quad (\text{A7})$$

$$\alpha_{11}^{78} = \frac{1}{2}\frac{a_3^4}{a_1^4}A_3^2 - 2Q_2^2A_1 + (2a_1^2Q_2^2 - 2a_1^2A_1 - 3a_3^2A_3)(-a_1^2A_{111} + \frac{1}{4}B_{111}) \\ + (2a_2^2Q_1^2 - 2a_2^2A_2 - 3a_3^2A_3)(\frac{1}{4}B_{112}) - \frac{5}{4}a_3^2A_3B_{113} + Q_1Q_2\left(1 - \frac{a_2^2}{a_1^2}\right)A_1 \\ - \frac{1}{4}Q_1Q_2^3 - a_2^2Q_1Q_2(3A_{11} + A_{12}) - \frac{1}{4}Q_2q_1 \quad (\text{A8})$$

¹¹For the terms in (11'), we checked our expressions by ensuring that $\nabla^2U = -4\pi G\rho$, (8') and the Newtonian equations hold. Furthermore, we tested the u 's by first ensuring that the moments \mathfrak{D}_i , \mathfrak{D}_{ijk} fulfil the appropriate Poisson equation and that the $\delta U^{(i)}$ of (69) from C67b agree with (70) and (71) from the same paper.

$$\begin{aligned}\alpha_{22}^{78} = & \frac{1}{2} \frac{a_3^4}{a_2^4} A_3^2 - 2Q_1^2 A_2 + (2a_1^2 Q_2^2 - 2a_1^2 A_1 - 3a_3^2 A_3) \left(\frac{1}{4} B_{122} \right) \\ & + (2a_2^2 Q_1^2 - 2a_2^2 A_2 - 3a_3^2 A_3) \left(-a_2^2 A_{222} + \frac{1}{4} B_{222} \right) - \frac{5}{4} a_3^2 A_3 B_{223} + Q_1^2 \left(1 - \frac{a_2^2}{a_1^2} \right) A_2\end{aligned}\quad (\text{A9})$$

$$\begin{aligned}\alpha_{33}^{78} = & \frac{1}{2} A_3^2 + \frac{1}{4} (2a_1^2 Q_2^2 - 2a_1^2 A_1 - 3a_3^2 A_3) B_{133} + \frac{1}{4} (2a_2^2 Q_1^2 - 2a_2^2 A_2 - 3a_3^2 A_3) B_{233} \\ & - \frac{5}{2} a_3^2 A_3 \left(-2a_3^2 A_{333} + \frac{1}{2} B_{333} \right)\end{aligned}\quad (\text{A10})$$

Let us summarize the arguments from above. We have checked all the equations in Part I of CE78 and find the analytic expressions to be free of error, except for the few minor points mentioned above. We have good reason to believe that both in that paper and here, Einstein's PN-equations are solved correctly including the PN-Bianchi identity. We obtain different numerical values for S_i which we suspect is related to a problem with the numerical evaluation of the α 's in CE78, though we cannot be certain that our matrices (M_{ij}) agree simply because their determinants vanish at the same point. The various tests of our α 's and the fact that we find the post-Newtonian Maclaurin spheroids in the axisymmetric case convince us that our values are correct.

A.1. The Solution at the Bifurcation Point

At the point $a_2 = a_1$, i.e. at the bifurcation point along the Maclaurin sequence, the following relations can be used to simplify the expressions for the α 's, where Ω refers to the angular velocity of the uniformly rotating Newtonian solution and has the same meaning as in C67b:

$$a_3^2 A_3 = a_1^4 A_{11} = a_1^2 (A_1 - B_{11}) = I - 2a_1^2 A_1 = a_1^2 \left(A_1 - \frac{1}{2} Q_1^2 \right), \quad \Omega^2 = 2B_{11} = Q_1^2. \quad (\text{A11})$$

Note that at this point, the α 's of C67b and C67a agree and we find

$$\begin{aligned}\alpha_1^{78} = & -\frac{a_3^4}{a_1^2} A_3^2 - 8a_1^2 B_{11} A_1 + 2IQ_1^2 - (2I + 3a_3^2 A_3) A_1 + (a_1^2 A_{11} - B_{11}) (2a_1^2 Q_1^2 - 2a_1^2 A_1 - 3a_3^2 A_3) + \frac{5}{2} a_3^2 A_3 B_{13} \\ = & -15a_1^2 A_1^2 - \frac{19}{4} a_1^2 Q_1^4 + 14a_1^2 A_1 Q_1^2 + \frac{5}{2} a_3^2 A_3 B_{13} \\ = & \alpha_1^{67} = \alpha_2^{78} = \alpha_2^{67}\end{aligned}\quad (\text{A12})$$

$$\begin{aligned}\alpha_3^{78} = & -a_3^2 A_3^2 - (2I + 3a_3^2 A_3) A_3 - B_{13} (2a_1^2 Q_1^2 - 2a_1^2 A_1 - 3a_3^2 A_3) + \frac{5}{2} a_3^2 A_3 (B_{33} - 2a_3^2 A_{33}) \\ = & a_1^2 (2Q_1^2 - 10A_1) A_3 + a_1^4 Q_1^2 A_{13} \\ = & \alpha_3^{67}\end{aligned}\quad (\text{A13})$$

$$\begin{aligned}\alpha_{12}^{78} = & \frac{a_3^4}{a_1^4} A_3^2 - 4Q_1^2 A_1 + a_1^2 \left(\frac{7}{2} Q_1^2 - 5A_1 \right) (-2a_1^2 A_{111} + B_{111}) - \frac{5}{2} a_3^2 A_3 B_{113} + \frac{1}{2} Q_1^4 + 8a_1^2 Q_1^2 A_{11} - \frac{1}{2} q_1 Q_1 \\ = & \alpha_{12}^{67} - \frac{1}{2} q_1 Q_1\end{aligned}\quad (\text{A14})$$

$$\begin{aligned}\alpha_{23}^{78} = & \frac{a_3^2}{a_1^2} A_3^2 - 2Q_1^2 A_3 + 4a_1^2 Q_1^2 A_{13} + a_1^2 \left(\frac{7}{2} Q_1^2 - 5A_1 \right) (B_{113} - a_1^2 A_{113}) - \frac{5}{2} a_3^2 A_3 (-2a_3^2 A_{133} + B_{133}) \\ = & \alpha_{23}^{67} = \alpha_{31}^{78} = \alpha_{31}^{67}\end{aligned}\quad (\text{A15})$$

$$\alpha_{11}^{78} = \alpha_{22}^{78} = \frac{1}{2} \alpha_{12}^{78} + \frac{1}{2} Q_1 q_1 = \alpha_{11}^{67} + \frac{1}{4} Q_1 q_1 \quad (\text{A16})$$

$$\begin{aligned}\alpha_{33}^{78} = & \frac{1}{2} A_3^2 + \frac{1}{2} a_1^2 \left(\frac{7}{2} Q_1^2 - 5A_1 \right) B_{133} - \frac{5}{2} a_3^2 A_3 (-2a_3^2 A_{333} + \frac{1}{2} B_{333}) \\ = & \alpha_{33}^{67}\end{aligned}\quad (\text{A17})$$

TABLE 1

THE NUMERICAL VALUES WE FIND FOR THE QUANTITIES LISTED IN TABLE 1 OF CHANDRASEKHAR & ELBERT (1978).

a_2/a_1	q_1	q_2	q_3	S_1	S_2	S_3	S_4	S_5	q	r_1	r_2	t_1	t_2
1.00	2.8272	-2.8272	0.0000	-0.3050	-0.3290	-0.0144	0.0574	0.2398	-2.7984	0.0288	-0.0288	0.0000	0.0000
0.99	2.8173	-2.8370	0.0211	-0.2944	-0.3323	-0.0132	0.0542	0.2481	-2.7984	0.0196	-0.0378	-0.0454	-0.0445
0.98	2.8073	-2.8470	0.0424	-0.2838	-0.3355	-0.0120	0.0508	0.2565	-2.7984	0.0101	-0.0466	-0.0923	-0.0887
0.97	2.7972	-2.8570	0.0639	-0.2733	-0.3387	-0.0107	0.0474	0.2648	-2.7984	0.0002	-0.0553	-0.1407	-0.1324
0.96	2.7869	-2.8671	0.0857	-0.2628	-0.3416	-0.0093	0.0438	0.2732	-2.7984	-0.0101	-0.0638	-0.1906	-0.1757
0.95	2.7766	-2.8774	0.1077	-0.2524	-0.3445	-0.0078	0.0401	0.2815	-2.7985	-0.0207	-0.0722	-0.2422	-0.2186
0.90	2.7231	-2.9297	0.2213	-0.2010	-0.3573	0.0010	0.0200	0.3223	-2.7989	-0.0804	-0.1121	-0.5287	-0.4282
0.85	2.6665	-2.9843	0.3414	-0.1509	-0.3675	0.0121	-0.0030	0.3610	-2.7999	-0.1528	-0.1486	-0.8741	-0.6316
0.80	2.6067	-3.0412	0.4691	-0.1019	-0.3754	0.0258	-0.0288	0.3949	-2.8020	-0.2413	-0.1818	-1.2996	-0.8317
0.75	2.5439	-3.1006	0.6052	-0.0538	-0.3813	0.0426	-0.0572	0.4198	-2.8064	-0.3506	-0.2120	-1.8363	-1.0329
0.70	2.4781	-3.1627	0.7511	-0.0064	-0.3856	0.0636	-0.0871	0.4280	-2.8152	-0.4872	-0.2398	-2.5324	-1.2409
0.66	2.4236	-3.2146	0.8758	0.0313	-0.3882	0.0845	-0.1106	0.4131	-2.8280	-0.6222	-0.2610	-3.2547	-1.4177
0.65	2.4098	-3.2279	0.9082	0.0408	-0.3888	0.0905	-0.1161	0.4049	-2.8324	-0.6603	-0.2662	-3.4656	-1.4642
0.64	2.3958	-3.2414	0.9412	0.0503	-0.3895	0.0968	-0.1215	0.3944	-2.8374	-0.7005	-0.2715	-3.6910	-1.5118
0.63	2.3818	-3.2550	0.9747	0.0598	-0.3901	0.1037	-0.1265	0.3814	-2.8432	-0.7427	-0.2768	-3.9323	-1.5607
0.62	2.3677	-3.2687	1.0087	0.0693	-0.3907	0.1109	-0.1311	0.3653	-2.8497	-0.7873	-0.2821	-4.1913	-1.6111
0.61	2.3536	-3.2826	1.0434	0.0789	-0.3914	0.1187	-0.1352	0.3459	-2.8573	-0.8343	-0.2876	-4.4697	-1.6632
0.60	2.3394	-3.2967	1.0786	0.0886	-0.3921	0.1271	-0.1388	0.3225	-2.8659	-0.8840	-0.2931	-4.7697	-1.7171
0.59	2.3252	-3.3109	1.1145	0.0983	-0.3929	0.1362	-0.1416	0.2947	-2.8758	-0.9367	-0.2989	-5.0939	-1.7732
0.58	2.3109	-3.3254	1.1510	0.1082	-0.3938	0.1460	-0.1436	0.2616	-2.8872	-0.9924	-0.3048	-5.4449	-1.8317
0.57	2.2967	-3.3400	1.1881	0.1182	-0.3948	0.1567	-0.1444	0.2224	-2.9004	-1.0516	-0.3110	-5.8262	-1.8929
0.56	2.2824	-3.3548	1.2260	0.1284	-0.3960	0.1685	-0.1439	0.1763	-2.9155	-1.1146	-0.3175	-6.2416	-1.9574
0.55	2.2681	-3.3699	1.2646	0.1387	-0.3973	0.1814	-0.1418	0.1219	-2.9330	-1.1818	-0.3244	-6.6957	-2.0254
0.50	2.1975	-3.4487	1.4690	0.1956	-0.4087	0.2720	-0.0912	-0.3361	-3.0751	-1.5988	-0.3691	-9.7775	-2.4444
0.45	2.1301	-3.5353	1.6959	0.2722	-0.4377	0.4537	0.1171	-1.4497	-3.4139	-2.2489	-0.4521	-15.4156	-3.1217
0.40	2.0694	-3.6327	1.9502	0.4253	-0.5330	0.9625	0.9405	-4.8952	-4.4794	-3.5956	-0.6789	-29.1295	-4.6607
0.35	2.0208	-3.7456	2.2386	1.6155	-1.4902	5.7347	10.0089	-37.9506	-15.1294	-13.4666	-2.7757	-137.5070	-16.8446
0.34	2.0132	-3.7707	2.3012	6.5850	-5.5917	26.0552	49.3174	-178.2659	-60.8513	-54.2058	-11.6728	-582.0542	-67.2855
0.33	2.0065	-3.7968	2.3655	-2.6453	2.0383	-11.7342	-23.8765	82.4680	24.2357	21.4267	4.8654	241.2314	26.2701
0.32	2.0007	-3.8240	2.4317	-1.0183	0.6975	-5.0896	-11.0416	36.4961	9.3003	8.0964	1.9536	94.8022	9.7077
0.30	1.9925	-3.8819	2.5703	-0.4078	0.1986	-2.6145	-6.2981	19.0909	3.7752	3.1511	0.8615	37.3393	3.3605
0.28	1.9892	-3.9456	2.7179	-0.2340	0.0578	-1.9180	-4.9749	13.7746	2.2579	1.8969	0.5449	17.7005	1.3877
0.25	1.9960	-4.0540	2.9584	-0.1417	-0.0205	-1.5433	-4.1994	9.8393	1.4994	1.8182	0.3549	0.1194	0.0075

B. Explicit expressions for S_1 , S_3 and r_1

At the point $a_2 = a_1$, the u 's from C67b and C67a are related by

$$u_{ij}^{(2)M} = -\frac{9}{13} \left(u_{ij}^{(3)} - 4u_{ij}^{(4)} - \frac{17a_1^2}{3a_3^2} u_{ij}^{(5)} \right) \Big|_{a_2=a_1}, \quad (\text{B1})$$

which follows from (119) of C67b.

Using these relations, those between the α 's and (11), (13) and (17), one finds that the third of equations (58) of CE78 becomes

$$0 = a_1^4 Q_{11}^{78} - a_1^2 a_3^2 Q_{13}^{78} + a_3^4 Q_{33}^{78} \\ \stackrel{a}{=} a_1^4 \alpha_{11}^{67} - a_1^2 a_3^2 \alpha_{13}^{67} + a_3^4 \alpha_{33}^{67} + \frac{a_1^4 Q_1}{2} r_1 + \frac{130}{9} a_1^2 a_3^2 A_3 S_3 - \frac{13}{9} \left(a_1^4 u_{11}^{(2)M} - a_1^2 a_3^2 u_{13}^{(2)M} + a_3^4 u_{33}^{(2)M} \right) S_3 \quad (\text{B2})$$

We thus have the solution

$$S_3 = \frac{9}{13} \frac{a_1^4 \alpha_{11}^{67} - a_1^2 a_3^2 \alpha_{13}^{67} + a_3^4 \alpha_{33}^{67} + a_1^4 Q_1 r_1 / 2}{a_1^4 u_{11}^{(2)M} - a_1^2 a_3^2 u_{13}^{(2)M} + a_3^4 u_{33}^{(2)M} - 10a_1^2 a_3^2 A_3}, \quad (\text{B3})$$

which agrees with (99) of C67a if we take (34) of this paper into account. In order to provide concise explicit formulæ, we again make use of the eccentricity

$$e = \sqrt{1 - a_3^2/a_1^2},$$

the quantity

$$C := 104e^6 - 444e^4 + 630e^2 - 245 \quad (\text{B4})$$

and recall that Q_1 is

$$Q_1 \stackrel{a}{=} -\sqrt{\frac{8e^2(1-e^2)}{3+8e^2-8e^4}}.$$

We now provide explicit expressions for S_1 , S_3 and r_1 . Note that the expressions for S_1 and S_3 can be obtained either as limiting values or by placing oneself directly on the point $a_2 = a_1$. On the other hand, r_1 can only be obtained by a limiting process. The formulæ read

$$S_1 \stackrel{a}{=} \frac{e}{2e^2-1} \left[\frac{-1}{26eC} (2864e^8 - 10128e^6 + 14712e^4 - 8120e^2 + 1365) Q_1^2 \right. \\ \left. + \frac{e}{3Q_1} w_1 + \frac{4e}{39CQ_1} (224e^6 - 840e^4 + 1170e^2 - 455) r_1 \right], \quad (\text{B5})$$

$$S_3 \stackrel{a}{=} \frac{36e^4}{65C} \left[\frac{(272e^4 - 244e^2 + 35) Q_1^2}{8e^2} - \frac{3e^2}{Q_1} r_1 \right], \quad (\text{B6})$$

$$r_1 \rightarrow \frac{-Q_1^3}{8e^2(2e^2+1)} (24e^4 - 12e^2 - 1) - \frac{7}{4} w_1. \quad (\text{B7})$$

In deriving these expressions, we have made use of the identities

$$a_3^2 \left(4A_{11} - \frac{2}{a_1^2} \right) - 4a_1^2 A_{11} + 3A_1 \Big|_{a_2=a_1} = 0, \quad (\text{B8})$$

$$3A_1^2 - 3A_1 - 4a_1^2 A_1 A_{11} + 5a_1^2 A_{11} - 2a_1^4 A_{11}^2 \Big|_{a_2=a_1} = 0. \quad (\text{B9})$$

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